Torus partition functions and spectra of gauged linear sigma models

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together with:

Fabian Ruehle
Hamburg
Overview of this talk

1. Motivation
2. Gauged linear sigma models
3. Full one-loop partition function
4. Modular invariance
5. Spectra
6. Conclusion
Calabi-Yau compactification with vector bundles

The basic requirement is that one obtains an effective 4D field theory with $\mathcal{N} = 1$ SUSY from the heterotic string:

Candelas, Horowitz, Strominger, Witten’85

\[ \mathcal{M}^{1,9} \to \mathcal{M}^{1,3} \times \mathcal{M}^6 \]

- a six dimensional Calabi-Yau manifold $\mathcal{M}^6$ with vanishing first Chern class

- a gauge background satisfying the Hermitean Yang–Mills equations characterized by a vector bundle
Toroidal orbifold geometries

The idea of orbifolds is that they are very simple geometries yet shared the main property of Calabi–Yau manifolds namely that only 4D $\mathcal{N} = 1$ SUSY survives.

Dixon, Harvey, Vafa, Witten’85, Ibanez, Mas, Nilles, Quevedo’87

Toroidal orbifolds are defined as

$$T^6 / G$$

with some six dimensional torus $T^6$ and a finite group $G$, like $\mathbb{Z}_N$
Comparison Calabi-Yaus and Orbifolds

Calabi-Yaus with vector bundles are SUGRA compactifications, hence constitute only a low energy limit of string theory:

- only access to the massless spectrum at best
- important worldsheet quantum effects may be missed: e.g. Wilson line consistency conditions [Błaszczyszyn et al. 2009]
- not every SUGRA background might have a string lift

Orbifolds which admit exact SCFT descriptions:

- full spectrum can be computed
- stringy conditions (e.g. modular invariance) can be checked
- but only account for special points in the moduli spaces of a minor set of all known CYs
Gauged linear sigma models (GLSMs) provide an approximate worldsheet description:

- **Compact CYs and vector bundles:** Witten’93, Distler’95
  - they define symplectic quotients related to projective spaces
  - their super potential lead to hyper surfaces constraints to describe complete intersection CYs with monad bundles

- **Non-compact CYs with line bundles** SGN’10
  - these in particular include non-compact orbifold resolutions

GLSMs are not SCFTs themselves, but are commonly believed to flow to genuine CFTs in the IR Silverstein, Witten’94
Recent developments:

Computation of GLSM partition functions

Using localization techniques it is possible to compute the full path integral for a GLSM on a given worldsheet: \cite{Witten88}.

Sphere partition functions \cite{BeniniCremonesi11, DoroudGomisLee11} which can be related to exact moduli Kähler potentials \cite{JockersKumarLapanMorrison11, GomisLee11}.

and torus partition functions to the elliptic genera \cite{GrassiPolicastroScheidegger07, BeniniEagerHoriTachikawa13, GaddeGukovPutrov12}.
(0,2) superfields

Chiral and Fermi matter superfields:

<table>
<thead>
<tr>
<th>Superfield</th>
<th>Chiral $z^a, z^a$</th>
<th>Fermi $\lambda^\alpha, \lambda^\alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(1)$</td>
<td>$q_a$</td>
<td>$q_a$</td>
</tr>
<tr>
<td>$U(1)_R$</td>
<td>$r_a$</td>
<td>$r_a^{-1}$</td>
</tr>
<tr>
<td>$U(1)_L$</td>
<td>$l_a$</td>
<td>$l_a$</td>
</tr>
</tbody>
</table>

Bosonic and fermionic gauge multiplets:

<table>
<thead>
<tr>
<th>Superfield</th>
<th>Bos. gauge $\mathcal{V}, A, a$</th>
<th>Fermi gauge $\Sigma, \varphi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U(1)_R$</td>
<td>0 -1 0</td>
<td>0 -1</td>
</tr>
<tr>
<td>$U(1)_L$</td>
<td>0 0 0</td>
<td>-1 -1</td>
</tr>
</tbody>
</table>
Worldsheet action

The (0,2) worldsheet action contains the following elements:

- **Gauge coupling**: \( \frac{1}{e^2} \)
  
  conformal limit \( e \to \infty \); free limit \( e \to 0 \).

- **Fayet-Iliopoulos (FI)-term**: \( \mathcal{W}_{\text{FI}} = \rho F \)
  
  encoding the Kahler parameters of the geometry

- **Superpotential**: \( \mu \mathcal{W}(\mathcal{Z}, \Lambda) \)
  
  encoding the hyper surface and monad bundle
We define the GLSM path integral:

\[
Z = \int \mathcal{D}(\mathcal{V}, \mathcal{A}) \mathcal{D}\Sigma \mathcal{D} \mathcal{Z} \mathcal{D} \Lambda \, e^{-S}.
\]

- on the WS torus with complex structure \( \tau \)
- with a gauge background determined by its holonomies

\[
\nu = \tau a + a', \quad a = \oint_{C_{\tau}} \alpha, \quad a' = \oint_{C_1} \alpha,
\]

- and the \( L \)- and \( R \)-symmetry holonomies encode the left- and right-moving spin-structures

\[
\nu_L = \frac{t'}{2} + \tau \frac{t}{2}, \quad \nu_R = \frac{s'}{2} + \tau \frac{s}{2}
\]

Benini, Eager, Hori, Tachikawa'13
Supersymmetric localization

Consider the deformed path integral: Witten’88

\[ Z(t) = \int D\Phi \ e^{-S-tS_{\text{exact}}}, \quad Q S = 0, \quad S_{\text{exact}} = Q U, \]

where \( S \) is the original action and \( S_{\text{exact}} \) a \( Q \)-exact modification.

\[ Q^2 = 0, \quad Q = \int d^2\sigma \psi(\sigma) \frac{\delta}{\delta Z(\sigma)} + \ldots \]

Then \( Z(t) \) is independent of the parameter \( t \):

\[ -\frac{dZ(t)}{dt} = \int D\Phi \ Q \left[ U e^{-S-tS_{\text{exact}}} \right] = 0 \]

Up to global subtleties on the worldsheet and in target space.
Supersymmetric localization

In particular, in the limit, $e \rightarrow 0$, the kinetic terms of the non-zero modes completely dominate over their interactions:

- The non-zero mode path integral reduces to a free theory one, giving rise to one-loop determinant factors

- The instanton contributions result from non-constant zero modes and e.g. encode Kaluza-Klein and winding numbers

The path integral becomes a finite-dimensional integral:

$$Z = \int d^2 \nu \int d\phi_0 \ Z_{\text{determinants}} \ Z_{\text{instantons}}$$
Chiral and Fermi superfield partition functions:

\[ Z_{\text{chiral}} = \frac{Z_d[v - e_d a_R]}{|Z_d[v']|^2}, \quad Z_{\text{fermi}} = Z_D^V = \frac{\theta[\frac{1}{2} e_D - V]}{\eta V} \]

with \( v = q \cdot a + l a_L + r a_R \) and \( V = Q \cdot a + L a_L + R a_R \)

Gauge and Fermi gauge multiplet partition functions:

\[ Z_{\text{bos. g.}} = Z_{\text{fer. g.}} = \frac{Z_D^D_{1}^{a_L}}{|Z_D^D_{1}^{a_R - a'_R}|^2} \]
Modular invariance

Modular transformations

The full one-loop partition function should be invariant under the modular transformation:

\[ T : \tau \to \tau + 1, \quad S : \tau \to -\frac{1}{\tau} \]
Conditions for modular invariance

This leads to conditions like:

\[ \frac{a_k}{2} (Q_k^T Q_l - q_k^T q_l) \equiv 0 \]

can be thought of as:

- **Orbifold CFT modular invariance condition:**

  In the sector \( \nu_R = 0 \Rightarrow a_k = 1/N \). With \( V = Q/N \) this gives:

  \[ \frac{N}{2} (V^2 - v^2) \equiv 0 \]

- **GLSM gauge anomaly conditions:**

  In the sectors \( \nu_R \neq 0 \Rightarrow a_k \in [0, 1] \), we find:

  \[ Q_k^T Q_l - q_k^T q_l = 0 \]
Determining the spectrum

The conventional massless spectrum computation of GLSMs makes use of $Q$-cohomology see e.g. Kachru, Witten’93, Adams’09, Bertolini, Melnikov, Plesser’14

We outline an alternative way to determine not only the massless but the full spectrum of GLSMs:

Perform $q, \bar{q}$-expansions of the full partition function:

$$Z = \sum q^{M_L^2} \bar{q}^{M_R^2} = \ldots$$
Mass formulas

For example the left–moving mass read

\[ M^2_L = \frac{1}{2} W^2_L + \frac{1}{2} P^2_L - 1 + \delta c + N_L \]

- where \( W_L \) encodes the instanton contributions
- and \( P_L \) is the so-called shifted momenta

\[ P_L = \left( P_D + \nu - \frac{1}{2} e_D \right) \bigg|_{P_{Db} - (a_R + \frac{1}{2} e_{Db})} \quad , \quad \rho_d \in \mathbb{Z}^d \]

The vacuum shift

\[ \delta c = \frac{1}{2} \tilde{\nu}^T (e_d - \tilde{\nu}) \frac{D_b + D_f}{8} + \frac{D_f}{2} \tilde{\nu} (1 - \tilde{\nu}) \]

These are generalizations of corresponding orbifold formulae.
Summary

GLSMs provide a worldsheet framework to study supersymmetric heterotic string on

- non-compact CYs with line bundles
- compact CY with non-trivial vector bundles

We used supersymmetric localization to determine their one-loop partition functions

Based on these results we:

- derived conditions to ensure one-loop modular invariance
- proposed a way to determine the full target space spectrum
Potential caveats in the localization

There are two situations in which localization might be obstructed because of global issues:

Target space boundary contributions:

The final step to show that the path integral did not depend on $t$ was:

$$-\frac{dZ(t)}{dt} = \int \mathcal{D}\Phi \int d^2\sigma \left\{ \psi(\sigma) \frac{\delta}{\delta Z(\sigma)} + \ldots \right\} \left[ U e^{-S-t S_{\text{exact}}} \right] \equiv 0$$

Hence this depends on

- the topology of the target space
- how the actions $S$ and $S_{\text{exact}}$ behave at these boundaries
Potential caveats in the localization

**Worldsheet boundary contributions:**

For the localization to work one needs $Q S = 0$.

However, in the worldsheet susy breaking sectors, i.e. $\nu_R \neq 0$, the super potential $\mathcal{W}$ is not Q-closed:

$$QS_{\mathcal{W}} = \int d^2 \sigma \, \partial \mathcal{W} \neq 0$$

because the boundary conditions of $\mathcal{W}$ are anti-periodic:

$$\mathcal{W}(\sigma + 1) = (-)^s \mathcal{W}(\sigma), \quad \mathcal{W}(\sigma - \tau) = (-)^{s'} \mathcal{W}(\sigma)$$

Hence in the following we assume that we are either in the sector $\nu_R = 0$ or the super potential is absent.