Heterotic Line Bundle Models on Smooth Calabi-Yau Manifolds

Andrei Constantin (University of Oxford)
Joint work with: Evgeny Buchbinder, Andre Lukas and Challenger Mishra

String Phenomenology 2014, Trieste
The Generation Problem
Hints from Heterotic Line Bundle Models

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A Wish List for (Heterotic) String Pheno

Four-dimensional EFT with:

- \( \mathcal{N} = 1 \) SUSY
- SM gauge interactions
- massless spectrum containing 3 chiral generations of quarks and leptons and no extra fields charged under the SM gauge group; uncharged fields (moduli) allowed for the moment
- massless spectrum containing Higgs doublets throughout the moduli space
- stable proton
- hierarchy of holomorphic Yukawa couplings consistent with a heavy top
- stable moduli; broken SUSY
- compute physical Yukawa couplings
Why 3 Generations?
**A Heterotic Setup**

In 10d, the heterotic string is specified by a metric and a non-abelian gauge field. To compactify: \((X, V)\).

**Constraints:**

\[
\text{ch}_2(V) - \text{ch}_2(TX) = [W]
\]

(Green-Schwarz anomaly cancellation)

\[
F_{ij} = F_{\bar{i}\bar{j}} = 0 \quad (V \text{ holomorphic})
\]

\[
g^{i\bar{j}}F_{i\bar{j}} = 0
\]

DUY theorem guarantees the HYM equation is satisfied provided that \(V\) is polystable and has slope zero.
A Heterotic Setup - continued

The simplest and best understood situation: \( X \) Calabi-Yau three-fold. (Complex, Kähler manifold with a no-where vanishing top form. Non-Kähler compactifications covered in Eirik Svanes’ talk.)

In this case, the possible bundles can be divided into two classes:

- \( V = TX \)
  - standard embedding: corresponds to \((2, 2)\) worldsheet susy

- \( V \neq TX \)
  - general embeddings: correspond to \((0, 2)\) worldsheet susy

In the following I will refer to the \( E_8 \times E_8 \) heterotic string.
In the standard embedding case \((V = TX)\), upon compactification one obtains \(E_6\) GUT models. The number of chiral families is given by:

\[
N_{\text{gen}} = -\frac{1}{2} \chi(X)
\]

For general embeddings involving bundles \(V\) with structure group \(SU(5)\), \(SU(4)\) or \(SU(3)\) (leading to \(SO(10)\), \(SU(5)\) and \(E_6\) GUTs):

\[
N_{\text{gen}} = -\text{ind}(V)
\]

Can we say more than this?
The Heterotic Line Bundle Setup

Simplest choice for $V$ (for, e.g. stability checks and cohomology computations): sum of line bundles

$$V = \bigoplus_{a=1}^{\text{rk}(V)} \mathcal{L}_a = \bigoplus_{a=1}^{\text{rk}(V)} \mathcal{O}(\vec{k}_a)$$

where $\vec{k}_a = c_1(\mathcal{L}_a)$.

$E_6$-models are obtained for $\text{rk}(V) = 3$, $SO(10)$-models for $\text{rk}(V) = 4$ and $SU(5)$-models for $\text{rk}(V) = 5$.

The (intermediate) GUT group contains 2, 3 and respectively 4 extra $U(1)$ symmetries. These are phono ok and can greatly constrain the superpotential.
The Heterotic Line Bundle Setup – continued

Topological constraints on $V$:

- $c_1(V) = 0$
- $c_2(TX) - c_2(V) \geq 0$
- $\text{ind}(V) = -3$

In addition, impose poly-stability and slope zero:

$$\mu(L_a) = \int_X c_1(L_a) \wedge J^2 = d_{ijk} \vec{k}^i_a \ t^j \ t^k = 0$$

simultaneously for all $a = 1, \ldots, \text{rk}(V)$

Result: intermediate GUT with a bunch of (effectively global) $U(1)$ symmetries, and 3 chiral families of matter.
Finiteness

Number of models

$k_{\text{max}}$

2 4 6 8 10

10 20 30 40 50

Number of models
A Theoretical Bound

\[ \sum_a k_a^T \tilde{G} k_a \leq |c_2(TX)| \]

\[ \tilde{G} = \kappa G / (6|t|) \]

\[ G_{ij} = \frac{1}{2 \text{Vol}(X)} \int_X J_i \wedge *J_j = -3 \left( \frac{\kappa_{ik}}{\kappa} - \frac{2\kappa_i \kappa_j}{3\kappa^2} \right) \]

where Vol(X) = \kappa/6 is the Calabi-Yau volume with respect to the Ricci-flat metric, \( \kappa = d_{ijk} t^i t^j t^k \), \( \kappa_i = d_{ijk} t^i t^j t^k \) and \( \kappa_{ij} = d_{ijk} t^k \).

\[ \sum_a |k_a|^2 \leq \frac{\text{num factor}}{\lambda_{\text{min}}} \]

To derive this, we used the slope-zero conditions and the bound on \( c_2(V) \) from the anomaly cancellation. The bound is not sensitive to the number of line bundles involved in the sum, nor to the index of \( V \). [Buchbinder, AC, Lukas]
Position in the Kähler Moduli Space

The constraints imposed by poly-stability $\mu(\mathcal{L}_1) = 0$ define a certain locus in the Kähler moduli space. The slope zero equations are homogeneous in the $t^i$. Thus $t^i \to \lambda t^i$ leaves this locus invariant.

There is a physically allowed region in the Kähler moduli space:

**Supergravity limit:** $t^i > 1$

**finiteness of low-energy coupling constants:** $\text{Vol}(X) \lesssim V_{\text{max}}$.

$$\text{Vol}(X) = \frac{1}{6} d_{ijk} t^i t^j t^k$$
SO(10) Models on the Tetraquadric CY

Calabi-Yau data:

\[ X = \begin{bmatrix} \mathbb{CP}^1 & 2 \\ \mathbb{CP}^1 & 2 \\ \mathbb{CP}^1 & 2 \\ \mathbb{CP}^1 & 2 \\ \end{bmatrix}^{4,68, -128} \]

\[
d_{ijk} = \int_X J_i \wedge J_j \wedge J_k = \begin{cases} 2 & \text{if } i \neq j, j \neq k \\ 0 & \text{otherwise} \end{cases}
\]

\[
\text{Vol}(X) = 2 (t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4)
\]

\[
c_2(TX) = (24, 24, 24, 24) .
\]
We scanned for line bundle sums that are poly-stable and have slope-zero in the ‘allowed’ region of the Kähler moduli space for different values of $V_{\text{max}}$.

We increased the range of integers defining $c_1(L_a)$ until no new models could be found. In all cases, the total number of models was finite.

We did this for many values of $\text{ind}(V)$, corresponding to $1 \leq N_{\text{gen}} \leq 80$. 
Statistics for \( \text{Vol}(X) = 100 \)
Statistics for $\text{Vol}(X) = 1000$
Statistics for Vol($X$) = 10,000
STATISTICS FOR $\text{Vol}(X) = 100,000$
Statistics for $\text{Vol}(X) = 100$ on TQ/$G_4$
Statistics for $\text{Vol}(X) = 100,000$ on TQ$/G_4$
CY volume and Physical Couplings

\[ \alpha_{\text{GUT}} = \frac{1}{2s} = \frac{1}{25} \]

\[ M_{\text{GUT}} = \times \frac{1}{2\pi l} \frac{1}{s^6} \simeq 2 \cdot 10^{16} \text{GeV} \]

\[ G_N = \frac{l^2}{8\pi} \frac{1}{s^{2/3}} \text{Vol}(X)^{-1/3} = \frac{1}{M_{\text{Pl}}^2} \]

\[ \text{Vol}(X) = \left( \frac{1}{32\pi^3} \right)^3 x^6 \frac{(2\alpha)^3}{M_{\text{GUT}}^6 G_N^3} \simeq x^6 10^5 \]
Conclusions

For a given manifold, there is an upper bound on the maximal number of generations that one can construct via a line bundle sum model. This bound can be controlled by imposing restrictions on Vol($X$), which can be computed in terms of physical couplings.

This gives a new relation between the number of generations, the topology of the compactification and physical constants in 4d.
Thank you!